XVI. *On the Contact of Conics with Surfaces.*

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It is well known that at every point of a surface two tangents, called principal tangents, may be drawn having three-pointic contact with the surface, *i.e.* having an intimacy exceeding by one degree that generally enjoyed by a straight line and a surface. The object of the present paper is to establish the corresponding theorem respecting tangent conics, viz. that "at every point of a surface ten conics may be drawn having six-pointic contact with the surface;" these may be called Principal Tangent Conics. In this investigation I have adopted a method analogous to that employed in my paper "On the Sextactic Points of a Plane Curve" (Philosophical Transactions, vol. clv. p. 653); and as I there, in the case of three variables, introduced a set of three arbitrary constants in order to comprise a group of expressions in a single formula, so here, in the case of four variables, I introduce with the same view two sets of four arbitrary constants. If these constants be represented by $\alpha, \beta, \gamma, \delta$; $\alpha', \beta', \gamma', \delta'$, I consider the conic of five-pointic contact of a section of the surface made by the plane $\varpi-k\varpi'=0$, where $\varpi=\alpha x+\beta y+\gamma z+\delta t$, and $\varpi'=\alpha' x+\beta' y+\gamma' z+\delta' t$, and k is indeterminate; and then I proceed to determine k, and thereby the azimuth of the plane about the line $\varpi=0$, $\varpi'=0$, so that the contact may be six-pointic. The formulæ thence arising turn out to be strictly analogous to those belonging to the case of three variables, except that the arbitrary quantities cannot in general be divided out from the final expression. In fact, it is the presence of these quantities which enables us to determine the position of the plane of section, and the equation whereby this is effected proves to be of the degree 10 in $\varpi : \varpi'=k$, and besides this of the degree $12n-27$ in the coordinates x, y, z, t, giving rise to the theorem above stated.

Beyond the question of the principal tangents, it has been shown by Clebsch and Salmon that on every surface U a curve may be drawn, at every point of which one of the principal tangents will have a four-pointic contact. And if n be the degree of U, that of the surface S intersecting U in the curve in question will be $11n-24$. Further, it has been shown that at a finite number of points the contact will be five-pointic. The number of these points has not yet been completely determined; but Clebsch has shown (Crelle, vol. lviii. p. 93) that it does not exceed $n(11n-24)(14n-30)$. Similarly it appears that on every surface a curve may be drawn, at every point of which one of the principal tangent conics has a seven-pointic contact, and that at a finite number of points the contact will become eight-pointic. But into the discussion of these latter problems I do not propose to enter in the present communication.

§ 1. *Conditions for a Sextactic Point.*

Let $U=0$ be the equation to the given surface, and $V=0$ that to the surface whose section by the plane $\varpi-k\varpi'=0$ is to have a six-pointic contact with the corresponding section of U at the point (x, y, z, t). Also, following the method of Professor CAYLEY (Philosophical Transactions, vol. cxlix. p. 371, and vol. clv. p. 545), let the coordinates of a point of U be considered as functions of a single parameter; then for the present purpose the coordinates of a point consecutive to (x, y, z, t) may be taken to be

$$x+dx+\tfrac{1}{2}d^2x+\tfrac{1}{6}d^3x+\tfrac{1}{24}d^4x+\tfrac{1}{120}d^5x,\ y+dy+\ldots,\ z+dz+\ldots,\ t+dt+\ldots; \quad . \quad . \quad (1)$$

and these values when substituted in U must satisfy the equation $U=0$. Then writing for shortness

$$\left.\begin{aligned}\partial_1&=dx\,\partial_x+dy\,\partial_y+dz\,\partial_z+dt\,\partial_t,\\ \partial_2&=d^2x\partial_x+d^2y\partial_y+d^2z\partial_z+d^2t\partial_t,\\ &\vdots \qquad \vdots \qquad \vdots\\ \partial_5&=d^5x\partial_x+d^5y\partial_y+d^5z\partial_z+d^5t\partial_t;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

substituting the values (1) in U, expanding as far as terms of the fifth degree, and arranging the result in lines of the degrees 0, 1, .. 5, respectively, we shall have

$$\left.\begin{aligned}0=&U\\ &+\partial_1U\\ &+\tfrac{1}{2}(\partial_1^2+\partial_2)U\\ &+\tfrac{1}{6}(\partial_1^3+3\partial_1\partial_2+\partial_3)U\\ &+\tfrac{1}{24}(\partial_1^4+6\partial_1^2\partial_2+4\partial_1\partial_3+\partial_4)U\\ &+\tfrac{1}{120}(\partial_1^5+10\partial_1^3\partial_2+10\partial_1^2\partial_3+15\partial_1\partial_2^2+5\partial_1\partial_4+10\partial_2\partial_3+\partial_5)U,\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

each line of which, being of an order different from the rest, must separately vanish.

Let us write, as usual,

$$\left.\begin{aligned}&\partial_xU=u\ , \quad \partial_yU=v\ , \quad \partial_zU=w\ , \quad \partial_tU=k,\\ &\partial_x^2U=u_1, \quad \partial_y^2U=v_1, \quad \partial_z^2U=w_1, \quad \partial_t^2U=k_1,\\ &\partial_y\partial_zU=u', \quad \partial_z\partial_xU=v'\ , \quad \partial_x\partial_yU=w',\\ &\partial_x\partial_tU=l'\ , \quad \partial_y\partial_tU=m', \quad \partial_z\partial_tU=n'.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then combining the equation $\partial_1U=0$ with the corresponding expression in V, viz. $\partial_1V=0$, we obtain the usual expressions for two-pointic contact, viz.

$$\frac{\partial_xV}{u}=\frac{\partial_yV}{v}=\frac{\partial_zV}{w}=\frac{\partial_tV}{k}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which, since U and V are both homogeneous in x, y, z, t, are equivalent to only two independent conditions. These conditions may be comprised in the single formula

$$\begin{vmatrix} \alpha, & \alpha', & u, & \partial_x V \\ \beta, & \beta', & v, & \partial_y V \\ \gamma, & \gamma', & w, & \partial_z V \\ \delta, & \delta', & k, & \partial_t V \end{vmatrix} = \square V = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $\alpha, \beta, \gamma, \delta$; $\alpha', \beta' \gamma', \delta'$ are arbitrary quantities to which any values may be given, provided only that all the determinants of the matrix

$$\begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix}$$

do not simultaneously vanish, since in that case the equation (6) would become nugatory. Comparing (6) with the equation $\partial_1 U = 0$, and observing that the differentials dx, dy, dz, dt may be replaced by the determinants

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ u, & v, & w, & k \end{matrix} \right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(the usual rule of signs being adopted, viz. the columns being taken in the cyclic order, the determinants are to have the signs $+, -, +, -$, respectively), it is clear that the equation $(\partial_1^2 + \partial_2)U = 0$, when combined with the corresponding equation in V, is equivalent to $\square^2 V = 0$. Similarly, the equation $(\partial_1^3 + 3\partial_1\partial_2 + \partial_3)U = 0$, combined with the corresponding equation in V, is equivalent to $\square^3 V = 0$; and so on. Hence the series of conditions comprised in (3) may be expressed as follows:

$$V = 0, \quad \square V = 0, \quad \square^2 V = 0, \quad \square^3 V = 0, \quad \square^4 V = 0, \quad \square^5 V = 0, \quad . \quad . \quad . \quad (8)$$

which correspond to equations (3) of my paper on Sextactic Points above quoted.

§ 2. *Preliminary Transformation.*

The next step is to effect a transformation of the first three equations of the system (8), corresponding to that given in § 1 of the same paper. As was stated in the introduction, the transformation does not throw out as factors any lineo-linear functions of the arbitrary quantities and the variables; but it reduces the expressions transformed to functions of such lineo-linear functions, viz. ϖ, and ϖ'.

Taking the columns of the matrix (7) two and two in the usual cyclic order, viz. β, γ; γ, α; α, β; α, δ; β, δ; γ, δ, and calling the determinants so formed a, b, c, f, g, h; *i. e.* writing

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix} \right\| = a, b, c, f, g, h; \quad . \quad . \quad . \quad . \quad (9)$$

and writing, further,

$$\left.\begin{aligned} \alpha x+\beta y+\gamma z+\delta t&=\varpi \\ \alpha' x+\beta' y+\gamma' z+\delta t'&=\varpi' \\ \alpha\varpi'-\alpha\varpi'&=\mathrm{A} \\ \beta'\varpi-\beta\varpi'&=\mathrm{B} \\ \gamma'\varpi-\gamma\varpi'&=\mathrm{C} \\ \delta'\varpi-\delta\varpi'&=\mathrm{D}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the quantities a, b, .. A, B, .. will be found to satisfy the following relations, useful in subsequent transformations, viz. :—

$$\left.\begin{aligned} &bz-cy-ft=\mathrm{A} \\ &cx-az-gt=\mathrm{B} \\ &ay-bx-ht=\mathrm{C} \\ &fx+gy+hz=\mathrm{D} \\ &\mathrm{A}x+\mathrm{B}y+cz+\mathrm{D}t=0 \\ &\mathrm{B}h-\mathrm{C}g+a\mathrm{D}=0 \\ &\mathrm{C}f-\mathrm{A}h+b\mathrm{D}=0 \\ &\mathrm{A}g-\mathrm{B}f+c\mathrm{D}=0 \\ &\mathrm{A}a+\mathrm{B}b+c\mathrm{C}=0 \\ &af+bg+ch=0 \\ &\varpi\partial_x\mathrm{A}=\alpha\mathrm{A}-\alpha\mathrm{A},\quad \varpi\partial_y\mathrm{A}=\alpha\mathrm{B}-\beta\mathrm{A},\quad \varpi\partial_z\mathrm{A}=\alpha\mathrm{C}-\gamma\mathrm{A},\quad \varpi\partial_t\mathrm{A}=\alpha\mathrm{D}-\delta\mathrm{A}, \\ &\varpi\partial_x\mathrm{B}=\beta\mathrm{A}-\alpha\mathrm{B},\quad \varpi\partial_y\mathrm{B}=\beta\mathrm{B}-\beta\mathrm{B},\quad \varpi\partial_z\mathrm{B}=\beta\mathrm{C}-\gamma\mathrm{B},\quad \varpi\partial_t\mathrm{B}=\beta\mathrm{D}-\delta\mathrm{B}, \\ &\varpi\partial_x\mathrm{C}=\gamma\mathrm{A}-\alpha\mathrm{C},\quad \varpi\partial_y\mathrm{C}=\gamma\mathrm{B}-\beta\mathrm{C},\quad \varpi\partial_z\mathrm{C}=\gamma\mathrm{C}-\gamma\mathrm{C},\quad \varpi\partial_t\mathrm{C}=\gamma\mathrm{D}-\delta\mathrm{C}, \\ &\varpi\partial_x\mathrm{D}=\delta\mathrm{A}-\alpha\mathrm{D},\quad \varpi\partial_y\mathrm{D}=\delta\mathrm{B}-\beta\mathrm{D},\quad \varpi\partial_z\mathrm{D}=\delta\mathrm{C}-\gamma\mathrm{D},\quad \varpi\partial_t\mathrm{D}=\delta\mathrm{D}-\delta\mathrm{D}. \end{aligned}\right\} \quad (11)$$

And in terms of this notation the developed form of $\square$ will be given by

$$-\square=(vh-wg+ka)\partial_x+(wf-uh+kb)\partial_y+(ug-vf+kc)\partial_z+(-ua-vb-wc)\partial_t. \quad (12)$$

This being premised, our first object is to investigate, as was done in the case of plane curves, an expression for $\square^2\mathrm{V}$, which in virtue of (12) will consist of two parts: first, terms of the form $(h\square v-g\square w+a\square k)\partial_x$, ..; and secondly, terms of the form $(vh-wg+ka)^2\partial_x^2$, .. Referring to (12), we have

$$\begin{aligned} h\square v-g\square w+a\square k=\;&\alpha,\ \alpha',\ u,\ hw'-gv'+al' \\ &\beta,\ \beta',\ v,\ hv_1-gu'+am' \\ &\gamma,\ \gamma',\ w,\ hu'-gw_1+an' \\ &\delta,\ \delta',\ k,\ hm'-gn'+ak_1 \end{aligned}$$

$$
=\begin{array}{cccccc}
. & . & . & \beta, & \gamma, & \delta \\
. & . & . & \beta', & \gamma', & \delta' \\
\alpha, & \alpha', & u, & w', & v', & l' \\
\beta, & \beta', & v, & v_1, & u', & m' \\
\gamma, & \gamma', & w, & u', & w_1, & n' \\
\delta, & \delta', & p, & m', & n', & k_1
\end{array}
=(n-1)^{-1}.\begin{array}{cccccc}
. & . & \alpha x-\varpi, & \beta, & \gamma, & \delta \\
. & . & \alpha' x-\varpi', & \beta', & \gamma', & \delta' \\
\alpha, & \alpha', & u_1x, & w', & v', & l' \\
\beta, & \beta', & w'x, & v_1, & u', & m' \\
\gamma, & \gamma', & v'x, & u', & w_1, & n' \\
\delta, & \delta', & l'x, & m', & n', & k_1,
\end{array}
$$

and writing

$$
\Phi=\begin{array}{cccccc}
u_1, & w', & v', & l', & \alpha, & \alpha' \\
w', & v_1, & u', & m', & \beta, & \beta' \\
v', & u', & w_1, & n', & \gamma, & \gamma' \\
l', & m', & n', & k_1, & \delta, & \delta' \\
\alpha, & \beta, & \gamma, & \delta, & . & . \\
\alpha', & \beta', & \gamma', & \delta', & . & .
\end{array}
\quad \dots\dots\dots \quad (13)
$$

we may deduce

$$
\left.\begin{array}{l}
h\square v-g\square w+a\square p=-\{(vh-wg+pa)\partial_x+\,.\,.\}(vh-wg+pa) \\
\qquad =(n-1)^{-1}\{x\Phi-\begin{array}{ccccc}
. & . & B, & C, & D\,\} \\
\alpha, & \alpha', & w', & v', & l' \\
\beta, & \beta', & v_1, & u', & m' \\
\gamma, & \gamma', & u', & w_1, & n' \\
\delta, & \delta', & l', & m', & k_1.
\end{array}
\end{array}\right\}
\quad \dots \quad (14)
$$

In the same way

$$
(-vh-wg+pa)^2=\begin{array}{cccccc}
. & . & . & \beta, & \gamma, & \delta \\
. & . & . & \beta', & \gamma', & \delta' \\
. & . & . & v, & w, & p \\
\beta, & \beta', & v, & v_1, & u', & m' \\
\gamma, & \gamma', & w, & u', & w_1, & n' \\
\delta, & \delta', & p, & m', & n', & k_1
\end{array}
=(n-1)^{-2}.\begin{array}{cccccc}
. & . & \alpha x-\varpi, & \beta, & \gamma, & \delta \\
. & . & \alpha' x-\varpi', & \beta', & \gamma', & \delta' \\
\alpha x-\varpi, & \alpha' x-\varpi, & u_1x^2, & w'x, & v'x, & l'x \\
\beta, & \beta', & w'x, & v_1, & u', & m' \\
\gamma, & \gamma', & v'x, & u', & w_1, & n' \\
\delta, & \delta', & l'x, & m', & n', & k_1
\end{array}
$$

$$
=(n-1)^{-2}\{x^2\Phi-2x\begin{array}{ccccc}
. & \alpha, & \beta, & \gamma, & \delta \\
. & \alpha', & \beta', & \gamma', & \delta' \\
B, & w', & v_1, & u', & m' \\
C, & v', & u', & w_1, & n' \\
D, & l', & m', & n', & k_1,
\end{array}
+\begin{array}{cccc}
. & B, & C, & D\} \\
B, & v_1, & u', & m' \\
C, & u', & w_1, & n' \\
D, & m', & n', & k
\end{array}
$$

or

$$(n-1)^2(vh-wg+ka)^2=x^2\Phi-2(n-1)x\Box(vh-wg+ka)-\left|\begin{array}{cccc} . & B & C & D \\ B & v_1 & u' & m' \\ C & u' & w_1 & n' \\ D & m' & n' & k_1 \end{array}\right| . \quad (15)$$

Similarly, it will be found that

$$\begin{aligned}&(n-1)^2(wf-uh+pb)(ug-vf+kc)\\ &=yz\Phi-2(n-1)\{y\Box(ug-vf+kc)+z\Box(wf-uh+kb)\}-\left|\begin{array}{cccc} . & A & C & D \\ A & u_1 & v' & l' \\ B & w' & u_1 & m' \\ D & l' & n' & k_1 \end{array}\right| ; \quad (16)\end{aligned}$$

so that, m being the degree of V, we shall have, on collecting all the terms of the forms (15) and (16), the following expression:—

$$\begin{aligned}&(n-1)^2\{(vh-wg+ka)^2\partial_x^2+\ldots+2(wf-uh-kb)(ug-vf+kc)\partial_y\partial_z+\ldots\}V\\ &=-2(n-1)(m-1)\{\Box(vh--wg+ka)\partial_xV+\ldots\}-\left|\begin{array}{cccc} . & B & C & D \\ B & v_1 & u' & m' \\ C & u' & w_1 & n' \\ D & m' & n' & k_1 \end{array}\right| \partial_x^2V-\ldots\end{aligned}$$

And if to each side of this equation we add

$$(n-1)^2\{(vh-wg+ka)^2\partial'_x+\ldots\}\{(vh-wg+ka)\partial_xV+\ldots\}=-(n-1)^2\{\partial_xV\Box(vh-wg+ka)+\ldots\},$$

in which the operations ∂'_x, .. affect the quantities u, .. only and not ∂_xV, ..; then we shall have the full expression for $\Box^2V$, viz.

$$-(n-1^2\left(1+\frac{2(m-1)}{n-1}\right)\{\partial_xV\Box(vh-wg+ka)+\ldots\}-\left|\begin{array}{cccc} . & B & C & D \\ B & v_1 & u' & m' \\ C & u' & w_1 & n' \\ D & m' & n' & k_1 \end{array}\right| \partial_x^2V-\ldots$$

Now referring to (5), and calling the ratios therein contained θ, and substituting θu for ∂_xV, θv for ∂_yV, . . , we have

$$u\Box(vh-wg+ka)+v\Box(wf-uh+kb)+w\Box(ug-vf+kc)+p\Box(-ua-vb-wc)$$

$$=\left|\begin{array}{ccccccc} . & . & . & \alpha & \beta & \gamma & \delta \\ . & . & . & \alpha' & \beta' & \gamma' & \delta' \\ . & . & . & u & v & w & k \\ \alpha & \alpha' & u & u_1 & w' & v' & l' \\ \beta & \beta' & v & w' & v_1 & u' & m' \\ \gamma & \gamma' & w & v' & u' & w_1 & n' \\ \delta & \delta' & p & l' & m' & u' & k_1 \end{array}\right| = n(-1)^{-2} \left|\begin{array}{ccccccc} . & . & -\varpi & \alpha & \beta & \gamma & \delta \\ . & . & -\varpi' & \alpha' & \beta' & \gamma' & \delta' \\ -\varpi & -\varpi' & . & . & . & . & . \\ \alpha & \alpha' & . & u_1 & w' & v' & l' \\ \beta & \beta' & . & w' & v_1 & u' & m' \\ \gamma & \gamma' & . & v' & u' & w_1 & n' \\ \delta & \delta' & . & l' & m' & n' & k_1 \end{array}\right| . = (n-1)^{-2} . \left|\begin{array}{ccccc} . & A & B & C & D \\ A & u_1 & w' & v' & l' \\ B & w' & v_1 & u' & m' \\ C & v' & u' & w_1 & n' \\ D & l' & m' & n' & k_1 \end{array}\right|$$

Further, if we agree upon the proper mode of development we may write

$$\begin{vmatrix} . & B & C & D \\ B & v_1 & u' & m' \\ C & u' & w_1 & u' \\ D & m' & n' & k_1 \end{vmatrix} \partial_x^2 V + \,.\,.+2 \begin{vmatrix} . & A & C & D \\ A & u_1 & v' & l' \\ B & w' & u' & m' \\ D & l' & n' & k_1 \end{vmatrix} \partial_y \partial_z V + \,.\,.$$

$$=\begin{vmatrix} u_1 & w' & v' & l' & A & \partial_x \\ w' & v_1 & u' & m' & B & \partial_y \\ v' & u' & w_1 & n' & C & \partial_z \\ l' & m' & n' & k_1 & D & \partial_t \\ A & B & C & D & . & . \\ \partial_x & \partial_y & \partial_z & \partial_t & . & . \end{vmatrix} V = \Delta V \text{ suppose} \quad . \quad . \quad . \quad . \quad (17)$$

which expression, by giving obvious values to $\overline{\mathfrak{A}}, \,.\,.\, \overline{\mathfrak{F}}, \,.\,.$, may be written thus:

$$\overline{\mathfrak{A}}\partial_x^2 V + \,.\,.+2\overline{\mathfrak{F}}\partial_y\partial_z V + \,.\,.=(\overline{\mathfrak{A}}, \overline{\mathfrak{B}}, \overline{\mathfrak{C}}, \overline{\mathfrak{D}}, \overline{\mathfrak{F}}, \overline{\mathfrak{G}}, \overline{\mathfrak{H}}, \overline{\mathfrak{L}}, \overline{\mathfrak{M}}, \overline{\mathfrak{N}})(\partial_x, \partial_y, \partial_z, \partial_t)^2 V. \quad . \quad (18)$$

Lastly, putting

$$H=\begin{vmatrix} u_1 & w' & v' & l' & A \\ w' & v_1 & u' & m' & B \\ v' & u' & w_1 & n' & C \\ l' & m' & n' & k_1 & D \\ A & B & C & D & . \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

the expression for $\square^2 V$ finally becomes

$$\Delta V - \left(1+\frac{2(m-1)}{n-1}\right)\theta H=0,$$

and consequently the system $V=0$, $\square V=0$, $\square^2 V=0$ may be replaced by

$$\frac{\partial_x V}{u}=\frac{\partial_y V}{v}=\frac{\partial_z V}{w}=\frac{\partial_t V}{k}=\left(1+\frac{2(m-1)}{n-1}\right)^{-1}\frac{\Delta V}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If in the foregoing expressions we put $\alpha'=0$, $\beta'=0$, $\gamma'=0$, $\delta'=1$, $\delta=0$, $t=0$, we shall have the case of plane curves, and as the last suppositions give $\varpi'=0$, $A=0$, $B=0$, $C=0$, $D=\alpha x+\beta y+\gamma z$, the expression (20) then reduces itself, as it should, to that given in equation (16) of the memoir above quoted.

§ 3. *Elimination of the Constants of the Quadric* V.

Before proceeding to the application of the formulæ (20) to the present problem, it will be convenient to premise that if ϕ, ψ be any two rational integral and homogeneous functions of x, y, z, t, the nature of the operation Δ is such that

$$\Delta\phi\psi=\psi\Delta\phi+\phi\Delta\psi+2(\overline{\mathfrak{A}}, \overline{\mathfrak{B}}, \overline{\mathfrak{C}}, \overline{\mathfrak{D}}, \overline{\mathfrak{F}}, \overline{\mathfrak{G}}, \overline{\mathfrak{H}}, \overline{\mathfrak{L}}, \overline{\mathfrak{M}}, \overline{\mathfrak{N}})(\partial_x\phi, \partial_y\phi, \partial_z\phi, \partial_t\phi)(\partial_x\psi, \partial_y\psi, \partial_z\psi, \partial_t\psi); \quad (21)$$

also that

$$\Delta U = 4H.$$

And further, if we write

$$\partial_x H = p,\ \partial_y H = q,\ \partial_z H = r,\ \partial_t H = s, \quad \ldots\ldots \quad (22)$$

and if we suppose that the first four columns in the four following determinants are the same as the first four in the expression for H given in (19), then

$$\Delta u = p - 2\begin{vmatrix} \ldots & 0 \\ \ldots & c \\ \ldots & -b \\ \ldots & f \\ \ldots & 0 \end{vmatrix},\ \Delta v = q - 2\begin{vmatrix} \ldots & -c \\ \ldots & 0 \\ \ldots & a \\ \ldots & g \\ \ldots & 0 \end{vmatrix},\ \Delta w = r - 2\begin{vmatrix} \ldots & b \\ \ldots & -a \\ \ldots & 0 \\ \ldots & h \\ \ldots & 0 \end{vmatrix},\ \Delta k = s - 2\begin{vmatrix} \ldots & -f \\ \ldots & -g \\ \ldots & -h \\ \ldots & 0 \\ \ldots & 0 \end{vmatrix}, \quad \ldots \quad (23)$$

from which it follows that

$$\left.\begin{aligned} \Delta(\ vh - wg + ka) &= \ qh - rg + sa \\ \Delta(\ wf - uh + kb) &= \ rf - ph + sb \\ \Delta(\ ug - vf + kc) &= \ pg - qf + sc \\ \Delta(-ua - vb - wc) &= -pa - qb - rc, \end{aligned}\right\} \quad \ldots\ldots \quad (24)$$

the outstanding terms which occur in (23) having cancelled one another.

Furthermore, applying the formula (21) to the following products, and bearing in mind that $\partial_x V, \partial_y V, \ldots$ are all linear in x, y, z, t, we have, by means of (24),

$$\left.\begin{aligned} \Delta\partial_x V(vh - wg + ka) &= \partial_x V(qh - rg + sa) + 2H(h\partial_x\partial_y V - g\partial_x\partial_z V + a\partial_z\partial_t V) \\ \Delta\partial_y V(wf - uh + kb) &= \partial_y V(rf - ph + sb) + 2H(f\partial_y\partial_z V - h\partial_x\partial_y V + b\partial_y\partial_t V), \\ \vdots \qquad\qquad &\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \end{aligned}\right\} \quad . \quad (25)$$

so that

$$\Delta\{\partial_x V(vh - wg + ka) + \partial_y V(wf - uh + kb) + \ldots\} = \begin{vmatrix} \alpha & \alpha' & p & \partial_x V \\ \beta & \beta' & q & \partial_y V \\ \gamma & \gamma' & r & \partial_z V \\ \delta & \delta' & s & \partial_t V \end{vmatrix}, \quad \ldots \quad (26)$$

the coefficients of 2H in (25) having cancelled one another.

This being premised, let us return to the equations (20). These equations being perfectly general, we may replace V by any other function we please. Hence if we replace V by $\square V$, the system resulting will be an equivalent for the system $\square V = 0$, $\square^2 V = 0$, $\square^3 V = 0$, viz. the second, third, and fourth of the system (8). Making this substitution in (20), and remembering that $\square V$ is of the degree n, so that the numerical factor $1 + \frac{2(m-1)}{n-1} = 3$, we have

$$\frac{1}{u}\partial_x \begin{array}{cccc} \alpha, & \alpha', & u, & \partial_x V \\ \beta & \beta' & v & \partial_y V \\ \gamma & \gamma' & w & \partial_z V \\ \delta & \delta' & k & \partial_t V \end{array} = \frac{1}{v}\partial_y \begin{array}{cccc} \alpha, & \alpha', & u, & \partial_x V \\ \beta & \beta' & v & \partial_y V \\ \gamma & \gamma' & w & \partial_z V \\ \delta & \delta' & k & \partial_t V \end{array} = \ldots = \frac{1}{3\mathrm{H}}\Delta \begin{array}{cccc} \alpha, & \alpha', & u, & \partial_x V \\ \beta & \beta' & v & \partial_y V \\ \gamma & \gamma' & w & \partial_z V \\ \delta & \delta' & k & \partial_t V \end{array};$$

or taking the first and last of these expressions, and substituting from (26),

$$\frac{3\mathrm{H}}{u}\left\{ \begin{array}{cccc} \alpha & \alpha' & u_1 & \partial_x V \\ \beta & \beta' & w' & \partial_y V \\ \gamma & \gamma' & v' & \partial_z V \\ \delta & \delta' & l' & \partial_t V \end{array} + \begin{array}{cccc} \alpha & \alpha' & u & \partial_x^2 V \\ \beta & \beta' & v & \partial_x\partial_y V \\ \gamma & \gamma' & w & \partial_x\partial_z V \\ \delta & \delta' & k & \partial_x\partial_t V \end{array}\right\} = \begin{array}{cccc} \alpha & \alpha' & p & \partial_x V \\ \beta & \beta' & q & \partial_y V \\ \gamma & \gamma' & r & \partial_z V \\ \delta & \delta' & s & \partial_t V \end{array};$$

or substituting θu for $\partial_x V, \ldots,$

$$\frac{3\mathrm{H}}{\theta u}\left\{ \begin{array}{cccc} \alpha & \alpha' & u_1 & u \\ \beta & \beta' & w' & v \\ \gamma & \gamma' & v' & w \\ \delta & \delta' & l' & k \end{array}\theta + \begin{array}{cccc} \alpha & \alpha' & u & \partial_x^2 V \\ \beta & \beta' & v & \partial_x\partial_y V \\ \gamma & \gamma' & w & \partial_x\partial_z V \\ \delta & \delta' & k & \partial_x\partial_t V \end{array}\right\} = \begin{array}{cccc} \alpha & \alpha' & p & u \\ \beta & \beta' & q & v \\ \gamma & \gamma' & r & w \\ \delta & \delta' & s & k \end{array},$$

or

$$\begin{array}{cccl} \alpha & \alpha' & u & (up-3\mathrm{H}u_1)\partial_x V + 3\mathrm{H}u\partial_x^2 V \\ \beta & \beta' & v & (uq-3\mathrm{H}w')\partial_x V + 3\mathrm{H}u\partial_x\partial_y V \\ \gamma & \gamma' & w & (ur-3\mathrm{H}v')\partial_x V + 3\mathrm{H}u\partial_x\partial_z V \\ \delta & \delta' & k & (us-3\mathrm{H}l')\partial_x V + 3\mathrm{H}u\partial_x\partial_t V \end{array} = 0; \quad \ldots \ldots \quad (27)$$

and if

$$\mathrm{V} = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{l}, \mathrm{m}, \mathrm{n})(x, y, z, t)^2, \quad \ldots \ldots \quad (28)$$

the expression (27) is equivalent to

$$\mathrm{a}\left\{ \begin{array}{cccl} \alpha & \alpha' & u & (up-3\mathrm{H}u_1)x + 3\mathrm{H}u \\ \beta & \beta' & v & (uq-3\mathrm{H}w')x \\ \gamma & \gamma' & w & (ur-3\mathrm{H}v')x \\ \delta & \delta' & k & (us-3\mathrm{H}l')x, \end{array}\right\} + \mathrm{h}\{..\} + \mathrm{g}\{..\} + \mathrm{l}\{..\} = 0 \quad . \quad . \quad (29)$$

which contains only the four coefficients a, h, g, l. In this expression the coefficient of

$$\mathrm{a} = \begin{array}{cccc} \varpi, & \varpi', & . & 2\mathrm{H}u \\ \beta, & \beta', & v, & uq-3\mathrm{H}w' \\ \gamma, & \gamma', & w, & ur-3\mathrm{H}v' \\ \delta, & \delta', & k, & us-3\mathrm{H}l'. \end{array} = \begin{array}{ccc} \mathrm{B}, & u, & uq-3\mathrm{H}w'-2\mathrm{H}u \\ \mathrm{C}, & w, & ur-3\mathrm{H}v' \\ \mathrm{D}, & k, & us-3\mathrm{H}l' \end{array} \begin{array}{ccc} \beta, & \beta', & v \\ \gamma, & \gamma', & w \\ \delta, & \delta', & k \end{array}$$

If, therefore,

$$\left.\begin{array}{l} X, Y, Z, T= \left\|\begin{array}{cccc} A, & B, & C, & D \\ p, & q, & r, & s \\ u, & v, & w, & k \end{array}\right\| \\ P_1, P_2, P_3, P_4= \left\|\begin{array}{cccc} A & B & C & D \\ u_1 & w' & v' & l' \\ u & v & w & k \end{array}\right\| +\tfrac{2}{3}u \left\|\begin{array}{cccc} \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \\ u & v & w & p \end{array}\right\| \end{array}\right\} \quad . \quad (30)$$

(29) will take the form

$$(uX-3HP_1)a+(uY-3HP_2)h+(uZ-3HP_3)g+(uT-3HP_4)l=0, \quad . \quad . \quad (31a)$$

to which might be added the analogous equations in h, b, f, m; g, f, c, n; l, m, n, d.

Another, and for some purposes a more convenient form may be given to these equations by the following transformation:

$H=(\mathfrak{A}', \ldots)(A, B, C, D)^2$, suppose;

$$\therefore \partial_x H=(\partial_x\mathfrak{A}', \ldots)(A, B, C, D)^2+\frac{2}{\varpi}(\mathfrak{A}', \ldots)(A, B, C, D)(\alpha A-\alpha A, \beta A-\alpha B, \gamma A-\alpha C, \delta A-\alpha D)$$

$$=(\partial_x\mathfrak{A}', \ldots)(A, B, C, D)^2+\frac{2A}{\varpi}(\mathfrak{A}', \ldots)(A, B, C, D)(\alpha, \beta, \gamma, \delta)-\frac{2\alpha}{\varpi}H.$$

If, therefore, henceforward p, q, r, s represent the differential coefficients of H upon the supposition that A, B, C, D are regarded as constants, we shall have

$$X=\begin{array}{ccc} B & C & D \\ q & r & s \\ v & w & k \end{array} +2H \begin{array}{ccc} \beta & \gamma & \delta \\ \beta' & \gamma' & \delta' \\ v & w & k \end{array}$$

$$P_1=\begin{array}{ccc} B & C & D \\ w' & v' & l' \\ v & w & k \end{array} +\tfrac{2}{3}u \begin{array}{ccc} \beta & \gamma & \delta \\ \beta' & \gamma' & \delta' \\ v & w & k, \end{array}$$

so that

$$uX-3HP_1=\begin{array}{cccc} . & B & C & D \\ 3H & q & r & s \\ u & w' & v' & l' \\ . & v & w & k \end{array} =(n-2)^{-1} \begin{array}{cccc} Ax & B & C & D \\ px & q & r & s \\ -u+u_1x & w' & v' & l' \\ ux & v & w & k; \end{array}$$

and if P, Q, R, S represent the determinants formed from the first three together with the fourth, the fifth, the sixth, and the seventh columns respectively of the following matrix:

$$\begin{array}{ccccccc} A & u & p & u_1 & w' & v' & l' \\ B & v & q & w' & v_1 & u' & m' \\ C & w & r & v' & u' & w_1 & n' \\ D & k & s & l' & m' & n' & k_1 \end{array}$$

and if, as was indicated above, H in the transformed expression is supposed to be differentiated without reference to A, B, C, D; then

$$\begin{aligned}(n-2)(uX-3HP_1)&=-uX+xP\\(n-2)(uY-3HP_2)&=-uY+yP\\(n-2)(uZ-3HP_3)&=-uZ+zP\\(n-2)(uT-3HP_4)&=-uT+tP,\end{aligned}$$

and the equations, of which (31 *a*) is one, take the form

$$\left.\begin{aligned}(uX-xP)\mathrm{a}+(uY-yP)\mathrm{h}+(uZ-zP)\mathrm{g}+(uT-tP)\mathrm{l}&=0\\(vX-xQ)\mathrm{h}+(vY-yQ)\mathrm{b}+(vZ-zQ)\mathrm{f}+(vT-tQ)\mathrm{m}&=0\\(wX-xR)\mathrm{g}+(wY-yR)\mathrm{f}+(wZ-zR)\mathrm{c}+(wT-tR)\mathrm{n}&=0\\(wX-xS)\mathrm{l}+(wY-yS)\mathrm{m}+(wZ-zS)\mathrm{n}+(wT-tS)\mathrm{d}&=0.\end{aligned}\right\}\quad . \; . \; . \quad (31\,b)$$

Representing any one of these equations, say, the first, by $W=0$, the equations $\Box^3V=0$, $\Box^4V=0$, $\Box^5V=0$ may be replaced by a system of the form (20); and writing them in the form $\partial_xW=\theta_1u$, $\partial_yW=\theta_1v$, .., where θ_1 is indeterminate, we may from the five equations so written eliminate the five quantities a, h, g, l, θ_1; and the resulting equation takes the form

$$\begin{vmatrix}\partial_x(uX-xP) & \partial_x(uY-yP) & \partial_x(uZ-zP) & \partial_x(uT-tP) & u\\ \partial_y(uX-xP) & \partial_y(uY-yP) & \partial_y(uZ-zP) & \partial_y(uT-tP) & v\\ \partial_z(uX-xP) & \partial_z(uY-yP) & \partial_z(uZ-zP) & \partial_z(uT-tP) & w\\ \partial_t(uX-xP) & \partial_t(uY-yP) & \partial_t(uZ-zP) & \partial_t(uT-tP) & k\\ \Delta(uX-xP) & \Delta(uY-yP) & \Delta(uZ-zP) & \Delta(uT-tP) & \nu H\end{vmatrix}=0 \quad . \; . \quad (32)$$

where ν is a numerical factor.

§ 4. *Determination of the extraneous factors.*

The degree of the equation (32) in its present form is $23n-32$; but it admits of reduction, in the first place, as follows. Since the equation

$$A(uX-xP)+B(uY-yP)+C(uZ-zP)+D(uT-tP)=0$$

is identically satisfied, we have by differentiation, and by substitution of the values of ∂_xA, .. from (11),

$$\begin{aligned}&A\partial_x(uX-xP)+B\partial_x(uY-yP)+C\partial_x(uZ-zP)+D\partial_x(uT-tP)\\&=-(uX-xP)\partial_xA-(uY-yP)\partial_xB-(uZ-zP)\partial_xC-(uT-tP)\partial_xD\\&=-\frac{1}{\varpi}\{(uX-xP)(\alpha A-\alpha A)+(uY-yP)(\alpha B-\beta A)+(uZ-zP)(\alpha C-\gamma A)+(uT-tP)(\alpha D-\delta A)\}\\&=\ \frac{A}{\varpi}\{\alpha(uX-xP)+\beta(uY-yP)+\gamma(uZ-zP)+\delta(uT-tP)\}\\&=\ \frac{A}{\varpi}K,\text{ suppose.}\end{aligned}$$

From this we may deduce the following system:

$$\left.\begin{array}{l} A\partial_x(uX-xP)+B\partial_x(uY-yP)+\ldots=A\frac{k}{\varpi} \\ A\partial_y(uX-xP)+B\partial_y(uY-yP)+\ldots=B\frac{k}{\varpi} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{array}\right\} \quad \ldots\ldots \quad (33)$$

Again, since $\Delta A=0$, $\Delta B=0$, $\Delta C=0$, $\Delta D=0$, it follows that

$$\begin{aligned} & A\Delta(uX-xP)+B\Delta(uY-yP)+C\Delta(uZ-zP)+D\Delta(uT-tP) \\ &=-\frac{2}{\varpi}\{(\mathfrak{A},\ldots)(\alpha A-\alpha A,\ \alpha B-\beta A,\ \alpha C-\gamma A,\ \alpha D-\delta A)(\partial_x,\partial_y,\partial_z,\partial_t)(uX-xP) \\ &\quad+(\mathfrak{A},\ldots)(\beta A-\alpha B,\ \beta B-\beta B,\ \beta C-\gamma B,\ \beta D-\delta B)(\partial_x,\partial_y,\partial_z,\partial_t)(uY-yP) \\ &\quad+(\mathfrak{A},\ldots)(\gamma A-\alpha C,\ \gamma B-\beta C,\ \gamma C-\gamma C,\ \gamma D-\delta C)(\partial_x,\partial_y,\partial_z,\partial_t)(uZ-zP) \\ &\quad+(\mathfrak{A},\ldots)(\delta A-\alpha D,\ \delta B-\beta D,\ \delta C-\gamma D,\ \delta D-\delta D)(\partial_x,\partial_y,\partial_z,\partial_t)(uT-tP)\}. \end{aligned}$$

But since

$$\left.\begin{array}{l} \mathfrak{A}A+\mathfrak{H}B+\mathfrak{G}C+\mathfrak{L}D=0 \\ \mathfrak{H}A+\mathfrak{B}B+\mathfrak{F}C+\mathfrak{M}D=0 \\ \mathfrak{G}A+\mathfrak{F}B+\mathfrak{C}C+\mathfrak{N}D=0 \\ \mathfrak{L}A+\mathfrak{M}B+\mathfrak{N}C+\mathfrak{D}D=0, \end{array}\right\} \quad \ldots\ldots\ldots \quad (34)$$

the expression above written reduces itself to

$$\begin{aligned} & \frac{2}{\varpi}\{A(\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)(\partial_x,\partial_y,\partial_z,\partial_t)(uX-xP) \\ &\quad+B(\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)(\partial_x,\partial_y,\partial_z,\partial_t)(uY-yP) \\ &\quad+C(\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)(\partial_x,\partial_y,\partial_z,\partial_t)(uZ-zP) \\ &\quad+D(\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)(\partial_x,\partial_y,\partial_z,\partial_t)(uT-tP)\} \\ &=\frac{2}{\varpi}\{\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)[(uX-xP)\partial_x A+(uY-yP)\partial_x B+\ldots \\ &\qquad\qquad (uX-xP)\partial_y A+(uY-yP)\partial_y B+\ldots \\ &\qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad] \\ &=-\frac{2k}{\varpi^2}(\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)(A,B,C,D) \\ &=0, \end{aligned}$$

i. e. $$A\Delta(uX-xP)+B\Delta(uY+yP)+C\Delta(uZ-zP)+D\Delta(uT-tP)=0. \quad \ldots \quad (35)$$

Hence, if we multiply the first three columns of (32) by A, B, C respectively, and add them to the fourth multiplied by D, the whole equation will be divisible by $\frac{K}{\varpi}$; and on the division being made the equation in question will take the form

$$\begin{array}{ccccc} \partial_x(uX-xP) & \partial_x(uY-yP) & \partial_x(uZ-zP) & A & u \\ \partial_y(uX-xP) & \partial_y(uY-yP) & \partial_y(uZ-zP) & B & v \\ \partial_z(uX-xP) & \partial_z(uY-yP) & \partial_z(uZ-zP) & C & w \\ \partial_t(uX-xP) & \partial_t(uY-yP) & \partial_t(uZ-zP) & D & k \\ \Delta(uX-xP) & \Delta(uY-yP) & \Delta(uZ-zP) & . & \nu H \end{array} =0 \quad \ldots \quad (36)$$

which is of the degree $18n-24$.

Next it may be shown that H is a factor of this expression. For when $H=0$ the following equations subsist,

$$\begin{aligned} Ax+By+Cz+Dt&=0\\ px+qy+rz+st&=0\\ ux+vy+wz+kt&=0, \end{aligned}$$

whence by elimination

$$\frac{X}{x}=\frac{Y}{y}=\frac{Z}{z}=\frac{T}{t}=\frac{P}{(n-1)u}; \qquad (37)$$

so that, omitting a numerical factor, $uX-xP$ will be replaced by xP, and (36) will then become

$$\begin{array}{ccccc} x\partial_x P+P & y\partial_x P & z\partial_x P & A & u\\ x\partial_y P & y\partial_y P+P & z\partial_y P & B & v\\ x\partial_z P & y\partial_z P & z\partial_z P+P & C & w\\ x\partial_t P & y\partial_t P & z\partial_t P & D & k\\ x\Delta P+2(\mathfrak{A}\partial_x+\ldots)P & y\Delta P+2(\mathfrak{H}\partial_x+\ldots)P & z\Delta P+2(\mathfrak{G}\partial_x+\ldots)P & . & . \end{array}$$

But by adding x (line 1)$+y$ (line 2)$+z$ (line 3) to t (line 4), the whole of line 4 will be divisible by P and a numerical factor; so that the expression becomes

$$\begin{array}{ccccc} x\partial_x P+P & y\partial_x P & z_x\partial P & A & u\\ x\partial_y P & y\partial_y P+P & z\partial_y P & B & v\\ x\partial_z P & y\partial_z P & z\partial_z P+P & C & w\\ x & y & z & . & .\\ x\Delta P+2(\mathfrak{A}\partial_x+\ldots)P & y\Delta P+2(\mathfrak{H}\partial_x+\ldots)P & z\Delta P+2(\mathfrak{G}\partial_x+\ldots)P & & . \end{array}$$

Again, subtracting $\partial_x P$ line 4 from line 1
$\partial_y P$ line 4 from line 2
$\partial_z P$ line 4 from line 3
ΔP line 4 from line 5,

and dividing throughout by 2, the expression is reduced to

$$\begin{array}{ccccc} P & . & . & A & u\\ . & P & . & B & v\\ . & . & P & C & w\\ x & y & z & . & .\\ (\mathfrak{A}\partial_x\ldots)P & (\mathfrak{H}\partial_x+\ldots)P & (\mathfrak{G}\partial_x+\ldots)P & . & . \end{array}$$

lastly, subtracting

A (column 1) +B (column 2) +C (column 3) from P (column 4),

and

u (column 1) $+v$ (column 2) $+w$ (column 3) from P (column 5), we have (remembering that $\mathrm{H}=0$)

$$\begin{array}{ccccc} \mathrm{P} & . & . & . & . \\ . & \mathrm{P} & . & . & . \\ . & . & \mathrm{P} & . & . \\ x & y & z & \mathrm{D}t & kt \\ (\mathfrak{A}\partial_x+\ldots)\mathrm{P} & (\mathfrak{H}\partial_x+\ldots)\mathrm{P} & (\mathfrak{G}\partial_x+\ldots)\mathrm{P} & \mathrm{D}(\mathfrak{L}\partial_x+\ldots)\mathrm{P} & k(\mathfrak{L}\partial_x+\ldots)\mathrm{P}\,; \end{array}$$

which vanishes identically. Hence H is a factor of (36); and on dividing it out the degree of the expression (36) is reduced to $15n-20$.

It remains now to be shown that u is likewise a factor of (36). Putting $u=0$, that equation becomes

$$\begin{array}{lccc} u_1\ \mathrm{X}-x\partial_x\mathrm{P}+\mathrm{P} & \ldots & \mathrm{A} & u \\ w'\ \mathrm{X}-x\partial_y\mathrm{P} & \ldots & \mathrm{B} & u \\ v'\ \mathrm{X}-x\partial_z\mathrm{P} & \ldots & \mathrm{C} & w \\ l'\ \mathrm{X}-x\partial_t\mathrm{P} & \ldots & \mathrm{D} & p \\ \Delta u\mathrm{X}-x\Delta\mathrm{P}+2\mathrm{H}\partial_x\mathrm{X}-2(\mathfrak{A}\partial_x+\ldots)\mathrm{P} & \ldots & 0 & \nu\mathrm{H} \end{array}$$

$$=\begin{array}{llccc} 1 & \mathrm{X} & \ldots & 0 & 0 \\ u_1 & x\partial_x\mathrm{P}+\mathrm{P} & \ldots & \mathrm{A} & u \\ w' & x\partial_y\mathrm{P} & \ldots & \mathrm{B} & v \\ v' & x\partial_z\mathrm{P} & \ldots & \mathrm{C} & w \\ l' & x\partial_t\mathrm{P} & \ldots & \mathrm{D} & p \\ \Delta u & z\Delta p+2\mathrm{H}\partial_x\mathrm{X}-2(\mathfrak{A}\partial_x+\ldots)\mathrm{P} & \ldots & 0 & \nu\mathrm{H}\,; \end{array}$$

and following a process similar to that adopted in the former transformation, this may be reduced to

$$\begin{array}{llccc} 1 & \mathrm{X} & \ldots & 0 & 0 \\ u_1 & \mathrm{P} & \ldots & \mathrm{A} & u \\ w' & . & \ldots & \mathrm{B} & v \\ v' & . & \ldots & \mathrm{C} & w \\ . & x & \ldots & 0 & 0 \\ \Delta u & 2\mathrm{H}\partial_x\mathrm{X}-2(\mathfrak{A}\partial_x+\ldots)\mathrm{P} & \ldots & 0 & \nu\mathrm{H}, \end{array}$$

and thence to

$$\begin{array}{llccc} t & t\mathrm{X}-x\mathrm{T} & \ldots & 0 & 0 \\ u_1 & \mathrm{P} & \ldots & 0 & 0 \\ w' & . & \ldots & 0 & 0 \\ v' & . & \ldots & 0 & 0 \\ . & x & \ldots & \mathrm{D}t & kt \\ \Delta u & 2\mathrm{H}\partial_x\mathrm{X}-2(\mathfrak{A}\partial_x+\ldots)\mathrm{P} & \ldots & \Theta & \Psi, \end{array}$$

Θ and Ψ being two functions, with the exact forms of which we are not concerned. The expression then takes the form of the product of two factors, viz. $(D\Psi-k\Theta)t$, and

$$\begin{vmatrix} t & tX-xT & tY-yT & tZ-zT \\ u_1 & P & . & . \\ w' & . & P & . \\ v' & . & . & P \end{vmatrix}$$

$$=P^2\{Pt-t(u_1X+w'Y+v'Z+l'T)+T(u_1x+w'y+v'z+l't)\}$$
$$=P^2t(P-P)$$
$$=0.$$

Hence u is a factor of (36). Now that equation as it stands is of the third degree in u; so that u being divided out it is reduced to the second degree, say, to the form $\lambda u^2+\mu u+\varrho=0$. Now referring to (31), and forming the equations in h, b, f, m; g, . .; l, . ., the equations corresponding to (36) may by a similar process be shown to be divisible by v, w, k; and those divisions having been effected, the equations in question will be reduced to the forms $\lambda v^2+\mu_1 v+\varrho_1=0$, . ., in which it is to be observed, from the symmetry of the expressions, that the coefficients of u^2, v^2, . . are the same. But as these equations (viz. those in u, v, w, k) all lead to the same result, namely the determination of the sextactic points, they can differ from one another only by factors. Hence, as in my memoir before quoted, equation (50), we must have identically

$$-\lambda=\frac{\mu u+\varrho}{u^2}=\frac{\mu_1 v+\varrho_1}{v^2}=\ldots \qquad (38)$$

which can hold good in general, only in virtue of μ being divisible by u, and ϱ by u^2, μ_1 being divisible by v_1 and ϱ_1 by v^2, and so on. Hence (36) is divisible not only by u but by u^3; and that division having been effected, the degree of (36) will be reduced to $12n-17$.

This completes the enumeration of the extraneous factors; but although the degree of the equation (36) cannot in general be depressed below $12n-17$, we have yet to show that, as stated in the introduction, the variables enter to the degree 10 in the form of lineo-linear functions of the arbitrary quantities. From what has gone before, it is clear that the quantities X, Y, Z, P, H involved in (36), are all functions of x, y, z, t, A, B, C, D only, that is to say, not of α, β, . . α', β', . . ., except in so far as they are included in A, B, C, D. It remains to be proved that this is still the case after the differentiations and operation Δ involved in (36) have been performed. For this purpose let ϕ represent any function of x, y, z, t, A, B, C, D, and let ∂'_x, . . indicate differentiation with respect to x, . . so far as they appear explicitly in ϕ, irrespectively of A, B, C, D, then

$$\partial_x\phi=\partial_A\phi\partial_x A+\partial_B\phi\partial_x B+\ldots+\partial'_x\phi;$$
$$\therefore\ \varpi\partial_x\phi=A(\alpha\partial_A+\beta\partial_B+\ldots)\phi-\alpha(A\partial_A+B\partial_B+\ldots)\phi+\varpi\partial'_x\phi;$$

and if ϕ be of the degree m in A, B, C, D, and $\nabla=\alpha\partial_{A}+\beta\partial_{B}+\ldots$, it follows that

$$\left.\begin{array}{l} \varpi\partial_x\phi=A\nabla\phi-m\alpha\phi+\varpi\partial'_x\phi \\ \varpi\partial_y\phi=B\nabla\phi-m\beta\phi+\varpi\partial'_y\phi \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \end{array}\right\} \quad \ldots \ldots \ldots \quad (39)$$

Again,

$$\begin{aligned} \varpi^2\partial_x^2\phi&=\varpi\partial_x\varpi\partial_x\phi-\alpha\varpi\partial_x\phi \\ &=(\varpi\partial_x-\alpha)\varpi\partial_x\phi, \end{aligned}$$

whence

$$\left.\begin{array}{l} \varpi^2\partial_x^2\phi\ \ =(A\nabla-\overline{m+1}\alpha+\varpi\partial'_x)(A\nabla-m\alpha+\varpi\partial'_x)\phi \\ \varpi^2\partial_y\partial_x\phi=(B\nabla-\overline{m+1}\beta+\varpi\partial'_y)(A\nabla-m\alpha+\varpi\partial'_x)\phi \\ \vdots \qquad \vdots \qquad \qquad \qquad \qquad \vdots \end{array}\right\} \quad \ldots \ldots \quad (40)$$

whence, again,

$$\begin{array}{l} \varpi^2(\mathfrak{A}\partial_x^2\ \ +\mathfrak{H}\partial_y\partial_x+\ldots)\phi=\{-(m+1(\mathfrak{A}\alpha+\mathfrak{H}\beta+\ldots)+\varpi(\mathfrak{A}\partial'_x+\mathfrak{H}\partial'_y+\ldots)\}(A\nabla-m\alpha+\varpi\partial'_x)\phi \\ \varpi^2(\mathfrak{H}\partial_x\partial_y+\mathfrak{B}\partial_y^2\ \ +\ldots)\phi=\{-(m+1)\mathfrak{H}\alpha+\mathfrak{B}\beta+\ldots)+\varpi(\mathfrak{H}\partial'_x+\mathfrak{B}\partial'_y+\ldots)\}(B\nabla-m\beta+\varpi\partial'_y)\phi\,; \\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \end{array}$$

and combining these expressions,

$$\varpi^2(\mathfrak{A},\ldots)(\partial_x,\ldots)^2\phi=m(m+1)(\mathfrak{A},\ldots)(\alpha,\ldots)^2\phi-(2m+1)\varpi(\mathfrak{A},\ldots)(\alpha,\ldots)(\partial'_x,\ldots)\phi+\varpi^2(\mathfrak{A},\ldots)(\partial'_x,\ldots)^2\phi.$$

Hence

$$\begin{array}{ccccc} \partial_x\phi & \partial_x\phi_1 & \partial_x\phi_2 & A & u \\ \partial_y\phi & \partial_y\phi_1 & \partial_y\phi_2 & B & v \\ \partial_z\phi & \partial_z\phi_1 & \partial_z\phi_2 & C & w \\ \partial_t\phi & \partial_t\phi_1 & \partial_t\phi_2 & D & k \\ \Delta\phi & \Delta\phi_1 & \Delta\phi_2 & . & \nu H \end{array}$$

$$=\varpi^{-4}\begin{array}{lccc} -m\alpha\phi+\varpi\partial'_x\phi & \ldots & A & u \\ -m\beta\phi+\varpi\partial'_y\phi & \ldots & B & v \\ -m\gamma\phi+\varpi\partial'_z\phi & \ldots & C & w \\ -m\delta\phi+\varpi\partial_t\phi & \ldots & D & k \\ \left(\begin{array}{l} \quad m(m+1)\,(\mathfrak{A},\ldots)(\alpha,\ldots)^2\phi \\ -(2m+1)\varpi(\mathfrak{A},\ldots)(\alpha,\ldots)(\partial'_x,\ldots)\phi \\ + \qquad \varpi^2(\mathfrak{A},\ldots)(\partial'_x,\ldots)^2\phi \end{array}\right) & \ldots & 0 & \nu H, \end{array}$$

which, by a method precisely similar to that followed above in investigating the extraneous factors, may be reduced to

$$\begin{array}{ccccc} \partial'_x\phi & \partial'_x\phi_1 & \partial'_x\phi_2 & A & u \\ \partial'_y\phi & \partial'_y\phi_1 & \partial'_y\phi_2 & B & v \\ \partial'_z\phi & \partial'_z\phi_1 & \partial'_z\phi_2 & C & w \\ \partial'_t\phi & \partial'_t\phi_1 & \partial'_t\phi_2 & D & k \\ \Delta\phi & \Delta\phi_1 & \Delta\phi_2 & . & \nu H, \end{array}$$

which is the same as if A, B, C, D had not been affected by the differentiations or by the operation Δ. The expression is therefore a function of x, y, z, t, A, B, C, D, and not of α, β, . . , α', β', . ., excepting so far as they appear in A, B, C, D. If, therefore, we put $\varphi=u\mathrm{X}-x\mathrm{P}$, $\varphi_1=u\mathrm{Y}-y\mathrm{P}$, $\varphi_2=u\mathrm{Z}-z\mathrm{P}$, we conclude that (36), when divested of its extraneous factors (say, Ω), is a function explicitly of the degree $12n-27$ in x, y, z, t, and of the degree 10 in A, B, C, D. But A, B, C, D are themselves linear homogeneous functions of ϖ, ϖ'; so that the expression in question may be regarded as explicitly of the degree $12n-27$ in x, y, z, t, and of the degree 10 in ϖ, ϖ'. This equation, solved for $k'=\varpi:\varpi'$, will consequently give ten positions of the cutting plane all passing through the point (x, y, z, t), for which the curve of section is sextactic at the point. Hence the theorem, "If L be any line through any point P of a surface, ten conics may be drawn in planes passing through L having six-pointic contact with the surface at the point."

It is to be observed that the plane of section is not necessarily normal; but if the line whose six coordinates are (a, b, c, f, g, h) be made to coincide with the normal at the point whose rectangular coordinates are x, y, z, it will follow that

$$a:b:c=u:v:w$$
$$\mathrm{A}u+\mathrm{B}v+\mathrm{C}w=0$$
$$fu+gv+hw=0,$$

with other relations which would abbreviate, although not essentially simplify, some previous expressions. The results of placing the line in the tangent plane are noticed below.

§ 5. *Note on tangents of more than two-pointic contact.*

If V, instead of being as hitherto quadric, be linear, we shall have the case of tangents to the curve of section of U with the plane $\varpi-k'\varpi'=0$. And if the contact be three-pointic, each of the ratios $\partial_x\mathrm{V}:u$, $\partial_y\mathrm{V}:v$, . . will be, in virtue of (20) of § 2, equal to $\Delta\mathrm{V}:\mathrm{H}$. But since V is linear $\Delta\mathrm{V}=0$, and consequently the condition for a tangent of three-pointic contact is $\mathrm{H}=0$. Now

$$\begin{aligned}\mathrm{H}&=(\mathfrak{A},\ldots)(\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D})^2\\&=(\mathfrak{A},\ldots)(\alpha'\varpi-\alpha\varpi',\ \beta'\varpi-\beta\varpi',\ \gamma'\varpi-\gamma\varpi',\ \delta'\varpi-\delta\varpi')^2\\&=\varpi'^2(\mathfrak{A},\ldots)(\alpha'k'-\alpha,\ \beta'k'-\beta,\ \gamma'k'-\gamma,\ \delta'k'-\delta)^2;\end{aligned}$$

so that the condition $\mathrm{H}=0$ may be written

$$(\mathfrak{A},\ldots)(\alpha',\beta',\gamma',\delta')^2k'^2-2(\mathfrak{A},\ldots)(\alpha',\beta',\gamma',\delta')(\alpha,\beta,\gamma,\delta)k+(\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)^2=0, \quad . \quad (41)$$

which will determine two values of k, and consequently two positions of the cutting plane for which the tangent line has three-pointic contact.

It may be noticed that in the solution of the equation above written there occurs the following expression:

$$(\mathfrak{A},\ldots)(\alpha',\beta',\gamma',\delta')^2\cdot(\mathfrak{A},\ldots)(\alpha,\beta,\gamma,\delta)^2-[(\mathfrak{A},\ldots)(\alpha',\beta',\gamma',\delta')(\alpha,\beta,\gamma,\delta)]^2.$$

But if H_0 represent the Hessian of U, then $\mathfrak{B}\mathfrak{C}-\mathfrak{F}^2=H_0(v_1w_1-u^2)$, . . ; and the expression in question will be equal to the product of the Hessian into

$$(v_1w_1-u'^2,\ w_1u_1-v'^2,\ ..\ v'w'-u_1u',\ ..)(a,\ b,\ c,\ f,\ g,\ h)^2,$$

where a, b, . . f, . . are the six coordinates of the line about which the cutting plane is supposed to revolve, as given in equations (9). The two positions of the cutting plane will coincide, if either the Hessian or this expression vanishes; the latter, regarded as a relation between the coordinates of the line, expresses the condition that the plane of section may contain both principal tangents, *i. e.* may be the tangent plane, as may be verified by the following considerations. If the plane of section coincide with the tangent plane, then

$$A:B:C:D=u:v:w:k;\ \text{or}\ \varpi\alpha'=\varpi'\alpha+\lambda u,\ \varpi\beta'=\varpi'\beta+\lambda v,\ ..,$$

which, being substituted in the expression in question, will cause it to vanish identically; or we may proceed otherwise, thus: regarding A, B, C, D as the constants of the cutting plane, the equation H=0 is

$$(\mathfrak{A},\ ..)(A,\ B,\ C,\ D)^2=0; \quad \ldots \quad (42)$$

the equation of the plane itself may be written

$$A\xi+B\eta+C\zeta+D\vartheta=0; \quad \ldots \quad (43)$$

and since this plane passes through the point (x, y, z, t), we have also

$$Ax+By+Cz+Dt=0; \quad \ldots \quad (44)$$

by means of which three equations the ratios A : B : C : D may be determined. By a process quoted by Professor CAYLEY (Quarterly Journal of Mathematics, vol. vii. p. 1), the solution of these equations depends upon the square root of the quantity

$$\begin{vmatrix} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L}, & x, & \xi \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M}, & y, & \eta \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N}, & z, & \zeta \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D}, & t, & \vartheta \\ x, & y, & z, & t & . & . \\ \xi, & \eta, & \zeta, & \vartheta & . & . \end{vmatrix} = H_0 \begin{vmatrix} u_1, & w', & v', & l', & x, & \xi \\ w', & v_1, & u', & m', & y, & \eta \\ v', & u', & w_1, & n', & z, & \zeta \\ l', & m', & n', & k_1, & t, & \vartheta \\ x, & y, & z, & t, & . & . \\ \xi, & \eta, & \zeta, & \vartheta, & . & . \end{vmatrix} \quad . . . \quad (45)$$

which is in fact identical with the expression (41).

The locus of the points for which one of the principal tangents meets a given line, say, the line $(a_1, b_1, c_1, f_1, g_1, h_1)$, will be found by eliminating A, B, C, D from the equations (42), (43), (44), combined with the following condition:

$$af_1+bf_1+ch_1+a_1f+b_1g+c_1h=0. \quad \ldots \quad (46)$$

But since the principal tangent is the intersection of the plane (A, B, C, D) with the tangent plane, we have

$$a,\ b,\ c,\ f,\ g,\ h, = \begin{Vmatrix} A, & B, & C, & D \\ u, & v, & w, & k \end{Vmatrix} \quad \ldots \quad (47)$$

Substituting these values, the condition above written becomes

$$\begin{aligned} &A(\ vh_1-wg_1+ka_1) \\ &+B(\ wf_1-uh_1+kb_1) \\ &+C(\ ug_1-vf_1+kc_1) \\ &+D(-ua_1-vb_1-wc_1)=0, \end{aligned}$$

and the ratios A : B : C : D will be expressed by the determinants of the matrix

$$\left\| \begin{matrix} vh_1-wg_1+ka_1, & wf_1-uh_1+kb_1, & ug_1-vf_1+kc_1, & -ua_1-vb_1-wc_1 \\ x & y & z & t \\ \xi & \eta & \zeta & \vartheta. \end{matrix} \right\|$$

But if by analogy with (11) we write

$$\begin{aligned} &b_1z-c_1y-f_1t=A_1, \\ &c_1x-a_1z-g_1t=B_1, \\ &a_1y-b_1x-h_1t=C_1, \\ &f_1x+g_1y+h_1z=D_1, \end{aligned}$$

it will be found that, putting

$$\begin{aligned} &\xi\ u+\eta\ v+\zeta w+\vartheta\ k=U_1, \\ &A_1\xi+B_1\eta+C_1\zeta+D_1\vartheta=E_1, \end{aligned}$$

the ratios A : B : C : D are equal to

$$A_1U_1-E_1u : B_1\,U_1-E_1v : C_1U_1-E_1w : D_1U_1-E_1u;$$

and consequently when these are substituted in H the terms having E_1 for a coefficient will vanish, and the equation of the locus resulting will be

$$(\mathfrak{A}, \ldots)(A_1, B_1, C_1, D_1)^2=0. \quad \ldots \ldots \ldots \quad (48)$$

To find the locus of points at which one of the principal tangents has a four-pointic contact with the surface, we must add to the equation H=0 the following, viz. $\square H=0$, which, as has been shown in a former part of this paper, may be replaced by any one of the group

$$\left\| \begin{matrix} A, & B, & C, & D \\ u, & v, & w, & k \\ \partial_xH, & \partial_yH, & \partial_zH, & \partial_tH \end{matrix} \right\| =0. \quad \ldots \ldots \ldots \quad (49)$$

But, remembering that p, q, r, s represent the differential coefficients of H with respect to x, y, z, t, on the supposition that A, B, C, D are constant, and writing

$$(\mathfrak{A}, \ldots)(A, B, C, D)(\alpha, \beta, \gamma, \delta)=H',$$

it is easy to deduce the following system:

$$\partial_xH=p+\frac{2A}{\varpi}H'$$

$$\partial_yH=q+\frac{2B}{\varpi}H'.$$

$$\vdots \quad \vdots \quad \vdots$$

But $\square \mathrm{H}=0$ is equivalent to the system $\partial_x\mathrm{H}=\theta_1 u$, $\partial_y\mathrm{H}=\theta_1 v$, .. *i. e.* to

$$\begin{aligned} \varpi p+2\mathrm{AH}'&=\theta_1 u \quad \text{or} \quad 2\mathrm{H}'\mathrm{A}=\theta_1 u-\varpi p \\ \varpi q+2\mathrm{BH}'&=\theta_1 v \qquad\quad 2\mathrm{H}'\mathrm{B}=\theta_1 v-\varpi q \\ \varpi r+2\mathrm{CH}'&=\theta_1 w \qquad\quad 2\mathrm{H}'\mathrm{C}=\theta_1 w-\varpi r \\ \varpi s+2\mathrm{DH}'&=\theta_1 k \qquad\quad 2\mathrm{H}'\mathrm{D}=\theta_1 k-\varpi s. \end{aligned}$$

Substituting these values of A, B, C, D, in the equation $\mathrm{H}=0$, the terms having θ_1 for a coefficient will vanish, and the equation will take the form

$$(\mathfrak{A}, \ldots)(p, q, r, s)^2=0. \qquad \ldots \ldots \ldots \quad (50)$$

But from what has gone before it is clear that H when resolved into its factors is of the form $\chi^2-\mathrm{H}_0\mu^2\psi^2$, where $\mathrm{H}_0\mu^2$ represents the expression (45); hence putting $\chi^2=\mu^2\psi^2\varphi^2$, $\frac{\mathrm{H}}{\mu^2\psi^2}=\varphi^2-\mathrm{H}_0$; and consequently since $\mathrm{H}=0$, $p=2\varphi\partial_x\varphi-\partial_x\mathrm{H}_0$, .. But $\varphi=\pm\sqrt{\mathrm{H}_0}$, the upper or lower sign being taken according as one or other principal tangent is the subject of consideration. Substituting then in the equation (50), we have

$$(\mathfrak{A}, \ldots)(2\sqrt{\mathrm{H}_0}\partial_x\varphi\pm\partial_x\mathrm{H}_0,\ 2\sqrt{\mathrm{H}_0}\partial_y\varphi\pm\partial_y\mathrm{H}_0, \ldots)^2=0.$$

But since we are seeking the condition under which either one or the other principal tangent may have four-pointic contact, the terms of ambiguous sign must disappear; and the condition required will take the form

$$4\mathrm{H}_0(\mathfrak{A}, \ldots)(\partial_x\varphi, \partial_y\varphi, \ldots)^2+(\mathfrak{A}, \ldots)(\partial_x\mathrm{H}_0, \partial_y\mathrm{H}_0, \ldots)^2=0,$$

which is of the degree $11n-24$, and may be compared with CLEBSCH'S form, viz. $(\mathfrak{A}, \ldots)(\partial_x\mathrm{H}_0, \ldots)^2-4\mathrm{H}_0\Phi$; but the comparison of the terms in φ and Φ appears difficult.

The additional condition for a five-pointic contact on the part of one of the principal tangents will be $\square^2\mathrm{H}=0$; or, having reference to (20) and to the consideration that $\mathrm{H}=0$, the condition will be $\Delta\mathrm{H}=0$. But the further discussion of this question I postpone to another occasion.